Lecture 11 Recall:

<u>Def.</u> The <u>(1,0)-tangent space $T_p^{1,0}M$</u> of M at $p \in M$ is

$$T_p^{1,0}M = T_p^{1,0}\mathbb{C}^n \cap \mathbb{C}T_pM = \left\{ Z = \sum_{j=1}^n a_j \frac{\partial}{\partial z_j} : \sum_{j=1}^n \frac{\partial \rho_k}{\partial z_j}(p) Z^j = 0,\ k = 1, \ldots, d \right\}.$$

- Linear algebra $\Rightarrow \dim_{\mathbb{C}} T_p^{1,0} = n - d'$, where $d' = \operatorname{rank}_{\mathbb{C}}(\partial\rho_1(p), \ldots, \partial\rho_d(p)) \le d$.

<u>Note</u>: ① $d\rho_1, \ldots, d\rho_d$ are lin. indep. / $\mathbb{R}$ but $\partial\rho_1, \ldots, \partial\rho_d$ need not be lin. indep. / $\mathbb{C}$.

② If $d = 1$, then $d\rho \ne 0 \Leftrightarrow \partial\rho \ne 0$. Thus, $\dim_{\mathbb{C}} T_p^{1,0}M = n-1$ for all $p \in M$. Real submflds of $\operatorname{codim}_{\mathbb{R}} = d = 1$ are called <u>real hypersurfaces</u>.

- <u>Ex1.</u> Consider the complex hyperplane $M = \{z_n = 0\} \cong \mathbb{C}^{n-1}$. As a real submfld, it is defined by $\operatorname{Re} z_n = x_n = 0$ and $\operatorname{Im} z_n = y_n = 0$

dx_n, dy_n are lin. indep. / $\mathbb{R} \Rightarrow d=2$, <u>but</u>

$x_n = \frac{1}{2}(z_n + z_n)$, $y_n = \frac{1}{2i}(z_n - z_n) \Rightarrow$

$\partial x_n = \frac{1}{2} dz_n$, $\partial y_n = \frac{1}{2i} dz_n$ <u>not</u> lin. indep.

over $\mathbb{C}$. Thus, $\dim_{\mathbb{C}} T^{1,0}_p M = n-1 = \dim_{\mathbb{C}} M$

This reflects the fact that M is a complex (holomorphic) manifold of $\dim_{\mathbb{C}} = n-1$.

<u>Ex2</u>. $M = \{ z_2 = |z_1|^2 \} \subseteq \mathbb{C}^2$. M is given as a real submfld by $\mathrm{Re}\, z_2 - |z_1|^2 = 0$, $\mathrm{Im}\, z_2 = 0$, and clearly $d\rho_1 \wedge d\rho_2 \neq 0$. But

$\partial \rho_1 = dz_2 - \bar{z}_1 dz_1$, $\partial \rho_2 = \frac{1}{2i} dz_2 \Rightarrow$

$$\mathrm{rank}_{\mathbb{C}} \{ \partial\rho_1, \partial\rho_2 \} = \begin{cases} 2, & z_1 \neq 0 \\ 1, & z_1 = 0. \end{cases}$$

Thus,
$$\dim_{\mathbb{C}} T^{1,0}_p M = \begin{cases} 0, & p \neq 0 \\ 1, & p = 0. \end{cases}$$

Def A real submfld $M \subseteq \mathbb{C}^n$ is called **CR** (a CR submfld) if $\dim_{\mathbb{C}} T_p M$ is constant over M. The **CR dim** of M is $\dim_{\mathbb{C}} T_p M$.

- Thus, real hypersurfaces are always CR.
- Complex manifolds, as in Ex 1, are CR.
- Ex 2 is **not** CR at 0.

Prop 1. If M is CR (w/ CR dim $= m$), then $T^{1,0}M = \coprod_{p \in M} T_p^{1,0} M$ is a subbundle of $T^{1,0}\mathbb{C}^n$, i.e., near each $p \in M$ $\exists$ $Z_1, \ldots, Z_m$ linearly indep / $\mathbb{C}$ $(1,0)$-vector fields (sections of $T^{1,0}\mathbb{C}^n$) that span $T_q^{1,0}M$ for q near p.

Pf. By def., there are $d' = n-m$ $\rho_1, \ldots, \rho_{d'}$ ($d' \leq d = \operatorname{codim}_{\mathbb{R}} M$) s.t. $\partial\rho_1, \ldots, \partial\rho_{d'}$ are lin. indep. at p and, hence, near p. Since $T^{1,0}\mathbb{C}^n$ is vector bundle, we may trivialize is $T^{1,0}\mathbb{C}^n \cong U \times \mathbb{C}^n$ near p, where U is small open nbhd of p.

Thus, we can identify $\eta_j(z) = \partial \rho_j(z)$, $j=1,..,d'$, with sections of the trivial bundle $U \times \mathbb{C}^n$. Standard linear algebra $\Rightarrow$ $\exists$ local frame $\xi_1 = (\xi_1^1, .., \xi_1^n), \ldots \xi_m \in U \times \mathbb{C}^n$ for the nullspace

$$\Big\{ \xi(z) \in U \times \mathbb{C}^n : \sum_{j=1}^{n} \eta_{k,j}(z)\, \xi^j = 0, \quad k=1,..,d' \Big\}.$$

<u>Ex.</u> Provide details for linear algebra above. ⊠

The fact that $T^{1,0}\mathbb{C}^n$ is bihol. invariant $\Rightarrow$ $T^{1,0}M$ is bihol. invariant.

Motivated by the def of holom. fcns and the chain rule discussion above, we have

<u>Def</u> ① If $M \subseteq \mathbb{C}^n$ is a CR submfld, then $f : M \to \mathbb{C}$ is <u>CR function</u> if

$$\bar{Z} f = 0, \quad \forall \text{ sections } \bar{Z} \text{ of } T^{0,1}M.$$

② A mapping $f: M \to M'$ (M, M' CR mflds) is <u>CR</u> if $f_*(T^{1,0}M) \subseteq T^{1,0}M$.

- As in the case of holom. mappings, if $M' \subseteq \mathbb{C}^N$, $f = (f_1, \dots, f_N)$ then $f: M \to M'$ is CR $\Leftrightarrow$ each component is CR.

A <u>CR diffeomorphism</u> is a diffeomorphism $f: M \to M'$ s.t. f^{-1} is CR.

<u>Note</u>: Existence of a CR diffeo $\Rightarrow$

$$\begin{cases} \dim_{\mathbb{R}} M = \dim_{\mathbb{R}} M' \ \underline{\text{and}} \\ \text{CR}\dim M = \text{CR}\dim M'. \end{cases}$$

Also, note that restriction to M of a bihólom. H is a CR diffeo onto $M' = H(M)$. We also have (by def.)

<u>Prop 2</u> $T^{1,0}M$ is CR invariant.